# Lecture 20 (2/25/22)

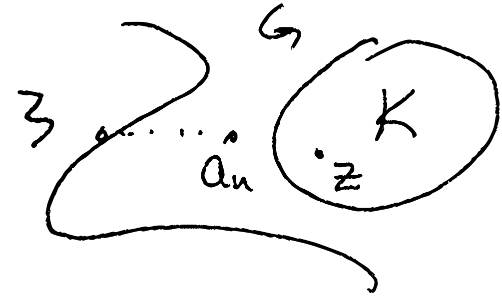
We now consider a construction thm for a general region $G \subseteq \mathbb{C}$. Let's first see what the idea is. Let $\{a_n\}_{n=1}^{\infty}$ be a seq. of points in $G$ that we would like to be the zero set of a function $f \in H(G)$. Thus, $\{a_n\}$ can only have limit points on $\partial_\infty G = \partial G \cup \{\infty\}$.

In case $G = \mathbb{C}$. $\partial_\infty G = \{\infty\}$, so $a_n \to \infty$.

For each $a_n$, we chose the factor $E_{p_n}(\varphi_n(z))$, where $\varphi_n(z) = z/a_n = 0$ for $z = a_n$ ($w=1$ is only zero of $E_{p_n}(w)$) and s.t. $\varphi_n(z) \to 0$ when $z \in K$, fixed compact, and $n \to \infty$. The latter was s.t. $\varphi_n(z) \in \mathbb{D} = \{|w|<1\}$ and we have $|E_{p_n}(\varphi_n(z)) - 1| \leq |\varphi_n(z)|^{p_n+1}$.

(by our key lemma). Then $\{p_n\}_{n=1}^{\infty}$ is chosen s.t. $\sum E_{p_n} - 1$ converges abs + unif on $K$.

<u>In case $G \subseteq \mathbb{C}$</u> is general and we have $a_n \to \zeta \in \partial G$, we can copy this




Take $\varphi_n(z) = \dfrac{\zeta - a_n}{z - a_n}$. We have

$\varphi_n(a_n) = 1$ and if $z \in K$, $|\varphi_n(z)| \leq$

$\frac{1}{\delta}|\zeta - a_n| \to 0$ as $n \to \infty$.

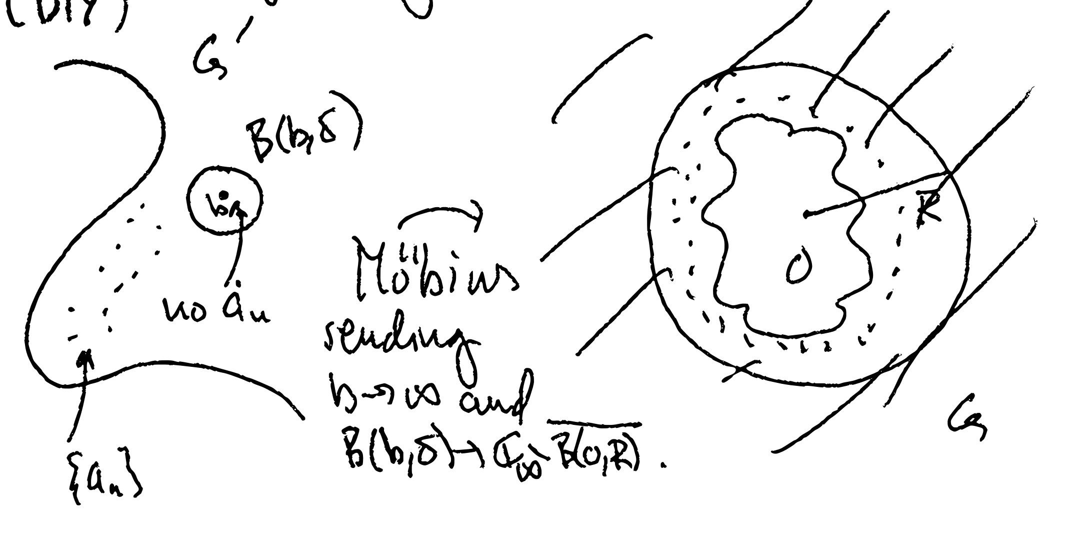
$\delta = d(K, \partial G)$

<u>Additional problem</u>. The seq. $\{a_n\}$ can have limitpoints all along $\partial G$ and at $\{\infty\}$.

Constr. Thm (General case). Let $\{a_n\}_{n=1}^{\infty}$ be a seq. of points in a region $G \subseteq \mathbb{C}$ without limit points in $G$. Then, $\exists$ $f \in H(G)$ whose zero set (w/ multipl.) is precisely $\{a_n\}_{n=1}^{\infty}$.

Pf. We make one simplifying assumption on $G$: $\mathbb{C} \setminus \overline{B(0,R)} \subseteq G$ and all $a_n \in G \cap B(0,R)$. We shall construct $f \in H(G)$ s.t. additionally $|f(z)| \le M$, $|z| \ge R' > R$.

If this can be done, the general case follows by using a Möbius transformation (DIY)



Thus, WLOG: $\partial_\infty G = \partial G \subseteq B(0,R)$.

For $a_n$, let $z_n \in \partial G$ s.t. $|z_n - a_n| \le \frac{3}{2}\delta_n$,

where $\delta_n = d(a_n, \partial G)$. Let $p_n = n-1$ (as one could do in Constr. Thm. for $G = \mathbb{C}$) and consider

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}\Big(\underbrace{\frac{z_n - a_n}{z - a_n}}_{=\varphi_n(z)}\Big).$$

This $f$ will satisfy the requirement (by previous results) if we show, ① $\forall K \subset\subset G$ $\sum_{n=1}^{\infty}(E_{p_n}(\varphi_n(z)) - 1)$ conv. abs. + unif. on $K$, and ② $|f(z)| \le M$ for $|z| \ge R'$.

①. Let $\delta = d(K, \partial G) > 0$. Then,

$$\varphi_n(z) = \left|\frac{z_n - a_n}{z - a_n}\right| \le \frac{3}{2}\frac{\delta_n}{\delta} \text{ for } z \in K.$$

Since $\delta_n \to 0$, $\exists N$ s.t. $\forall n \ge N$

$|\varphi_n(z)| \le \frac{1}{2} \Rightarrow$

$$|E_{p_n}(\varphi_n(z)) - 1| \le \left(\frac{1}{2}\right)^{p_n+1} = \frac{1}{2^n}.$$

This proves (by Weierstrass M-test) that $\sum_{n=1}^{\infty} E_{p_n}(z) - 1$ conv. abs. + unif. on K.

(2) We also note $|\varphi_n(z)| \le \frac{2R}{|z|-R}$

Recall that $\exists\, 0<\mu<1$, s.t. for $|w|<\mu$

$$\frac{1}{2}|w| \le |\log(1+w)| \le \frac{3}{2}|w|.$$

Thus, $\exists\, R'>R$ s.t. $|\varphi_n(z)| < \mu$, $|z| \ge R'$,

$\Rightarrow |E_{p_n}(\varphi_n(z)) - 1| \le \mu^n \Rightarrow$ ( $\operatorname{Re} E_{p_n} \circ \varphi_n > 0$ and) $|\operatorname{Log} E_{p_n}(\varphi_n(z))| \le \frac{3}{2} |E_{p_n}(\varphi_n(z)) - 1|$

$\le \frac{3}{2}\mu^n \Rightarrow$

$$\left|\prod_{n=1}^{\infty} E_{p_n}(\varphi_n(z))\right| \le e^{\sum_{n=1}^{\infty} |\mathrm{Log}\, E_{p_n}(\varphi_n(z))|}$$

$$\le e^{\frac{3}{2}\sum_{n=1}^{\infty} \mu^n} = M. \qquad \boxtimes$$

Reveals structure of $\mathcal{M}(G)$.

<u>Cor.</u> Let $f \in \mathcal{M}(G)$. Then $\exists\, h, g \in H(G)$ s.t. $f = h/g$.

<u>Pf.</u> Construct $h, g$ using Constr. Thm from zero set and pole set of $f$. $\boxtimes$